$\Rightarrow |f_m(x) - f_n(x)| < \epsilon/2 + \epsilon/2 = \epsilon$

$\Rightarrow |f_m(x) - f_n(x)| < \epsilon \quad \forall\ m>N,\ n>N$ and $\forall\ x \in S$

$\therefore$ Cauchy condition is satisfied.

Sufficient part:

Assume that, for every $\epsilon>0$ there exist N such that

$$|f_m(x) - f_n(x)| < \epsilon \quad \forall\ m>N,\ n>N \text{ and } \forall\ x \in S \quad \longrightarrow (2)$$

T.P.T There exists a function f such that,

$f_n \to f$ uniformly on S.

Since condition (2) exists, for each $x \in S$, the sequence $\{f_n(x)\}$ converges.

Let $f(x) = \lim_{n \to \infty} f_n(x)$ if $x \in S$

we must S.T $f_n \to f$ uniformly on S.

(2) $\Rightarrow$ for any given $\epsilon>0$, we can choose N so that $n>N$ implies

$$|f_n(x) - f_{n+k}(x)| < \epsilon/2 \quad \forall\ k=1,2 \ldots \text{ and } \forall\ x \in S \quad \longrightarrow (3)$$

Consider,

$$\lim_{k \to \infty} |f_n(x) - f_{n+k}(x)| = |\lim_{k \to \infty} f_n(x) - \lim_{k \to \infty} f_{n+k}(x)|$$

$$\leq |f_n(x) - f(x)|$$

$$\leq \epsilon/2$$

(3) $\Rightarrow |f_n(x) - f(x)|$

$\Rightarrow |f_n(x) - f(x)| < \epsilon \quad \forall\ n>N$ and $\forall\ x \in S$

$\therefore f_n \to f$ uniformly on S

Hence the proof.

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9.6 Uniform Convergence of Infinite Series of Functions

Def : 9.4

Given a sequence $\{f_n\}$ of functions defined on a set S. For each x in S, let,

$$s_n(x) = \sum_{k=1}^{n} f_k(x), \quad n=1,2\ldots$$

If there exists a function f such that $s_n \to f$ uniformly on S, we say the series $\sum f_n(x)$ converges uniformly on S and we write

$$\sum_{n=1}^{\infty} f_n(x) = f(x) \text{ uniformly on } S.$$

Theorem 9.5 Cauchy condition for Uniform Convergence of Series.

The infinite series $\sum f_n(x)$ converges uniformly on S if and only if for every $\epsilon > 0$ there is an N such that $n > N$ implies

$$\left| \sum_{k=n+1}^{n+p} f_k(x) \right| < \epsilon \quad \text{for each } p=1,2,\ldots \text{ and every}$$

x in S.

Proof:

Consider the partial sum of the series $\sum f_n(x)$,

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \quad ; \forall x \in S \quad , n=1,2\ldots$$

By the definition of uniform convergence of the series $\sum f_n(x)$ converges uniformly to f on S if $\{s_n(x)\}$ converges to f uniformly.

By using cauchy condition for the uniform convergence of the sequence we have,

$\{s_n\}$ converges to f uniformly iff

For any given $\epsilon > 0$, there exists N such that

$m > N$ and $n > N$

Implies $|s_m(x) - s_n(x)| < \epsilon \quad \forall x \in S$

Choose $m = n+p$, then we have

$\sum f_n(x)$ converges uniformly iff

$|s_{n+p}(x) - s_n(x)| < \epsilon \quad \forall x \in S$ and $p = 1, 2, \ldots$ and $n > N$

ie) $\left| \sum_{k=1}^{n+p} f_k(x) - \sum_{k=1}^{n} f_k(x) \right| < \epsilon \quad \forall x \in S$ and $n > N$ and $p = 1, 2, \ldots$

ie) $\left| \sum_{k=n+1}^{n+p} f_k(x) \right| < \epsilon \quad \forall x \in S$ and $p = 1, 2, \ldots$ $n > N$.

Hence $\sum f_n(x)$ convergences uniformly on S iff for every $\epsilon > 0$ there exists N such that

$\left| \sum_{k=n+1}^{n+p} f_k(x) \right| < \epsilon \quad \forall x \in S$, $p = 1, 2, \ldots$ and $n > N$.

Theorem : 9.6

State and prove Weierstrass M - test

Statement:

Let $\{M_n\}$ be a sequence of non negative number such that

$0 \leq |f_n(x)| \leq M_n$, for $n = 1, 2, \ldots$ and $\forall x \in S$

then $\sum f_n(x)$ converges uniformly on S if $\sum M_n$ converges.

proof:

given,

(i) $\{M_n\}$ is a sequence of non negative terms such that

$0 \leq |f_n(x)| \leq M_n$ for $n = 1, 2, \ldots$ and $\forall x \in S$ $\rightarrow (1)$

(ii) $\sum M_n$ converges $\rightarrow (2)$

T.P.T $\sum f_n(x)$ converges uniformly on S.

T.P.T for every $\epsilon > 0$ there exists an integer N such that $n > N$ implies $\left| \sum_{k=n+1}^{n+p} f_k(x) \right| < \epsilon$ for each $p = 1, 2, \ldots$ $\forall x \in S$

(By Cauchy condition for uniform convergence of infinite series)

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Consider: $\left|\sum_{k=n+1}^{n+p} f_k(x)\right| \le \sum_{k=n+1}^{n+p} |f_k(x)|$

$\le \sum_{k=n+1}^{n+p} M_k \quad \forall\, x \in S \longrightarrow (3)$

given that $\sum M_n$ converges

By cauchy condition for series

The series $\sum M_n$ converges iff for every $\epsilon > 0$ there exists an integer N such that

$$\left|\sum_{k=n+1}^{n+p} M_k\right| < \epsilon, \quad p = 1, 2, \ldots \text{ and } \forall\, n > N$$

for every $\epsilon > 0$

$$(3) \Rightarrow \left|\sum_{k=n+1}^{n+p} f_k(x)\right| \le \sum_{k=n+1}^{n+p} M_k$$

$< \epsilon \;\; \forall\, x \in S, \; p = 1, 2 \ldots$ and $\forall\, n > N$

$$\Rightarrow \left|\sum_{k=n+1}^{n+p} f_k(x)\right| < \epsilon \;\; \forall\, x \in S, \; p = 1, 2 \ldots \text{ and } n > N$$

$\therefore \sum f_n(x)$ converges uniformly on S.

Theorem: 9.7

Assume that $\sum f_n(x) = f(x)$ uniformly on S. If each f_n is continuous at a point x_0 of S then f is also continuous at x_0

Proof: given, (i) $\sum f_n(x) = f(x)$ uniformly on S

(ii) Each f_n is continuous at a point x_0 of S

T.P.T f is continuous at x_0

Define for each $x \in S$ and for each $n = 1, 2 \ldots$

$$s_n(x) = \sum_{k=1}^{n} f_k(x)$$

$\because$ each f_n is continuous at $x_0 \in S$

s_n is continuous at $x_0 \;\forall n \longrightarrow (1)$

$\because \sum f_n(x) = f(x)$ uniformly on S, there exists a function f such that

S_n converges to f uniformly on S ——→ (2)

Since each S_n is continuous at x_0 and $S_n \to f$ uniformly on S f is continuous at x_0

Note :

For each $x \in [a,b]$

$$\lim_{n\to\infty} \int_a^x f_n(t)\, d\alpha(t) = \int_a^x \lim_{n\to\infty} f_n(t)\, d\alpha(t)$$

Uniform convergence of and Riemann - Stieltjes Integration.

Theorem 9.8 :

Let α be of bounded variation on $[a,b]$. Assume that each term of the sequence $\{f_n\}$ is a real valued function such that $f_n \in R(\alpha)$ on $[a,b]$ for each $n=1,2\ldots$ Assume that $f_n \to f$ uniformly on $[a,b]$ and define $g_n(x) = \int_a^x f_n(t)\, d\alpha(t)$ if $x \in [a,b]$ $n = 1,2\ldots\ldots$ then we have

a) $f \in R(\alpha)$ on $[a,b]$

b) $g_n \to g$ uniformly on $[a,b]$. where $g(x) = \int_a^x f(t)\, d\alpha(t)$

Note: The conclusion implies that, for each x in $[a,b]$, we can write

$$\lim_{n\to\infty} \int_a^x f_n(t)\, d\alpha(t) = \int_a^x \lim_{n\to\infty} f_n(t)\, d\alpha(t)$$

This property is often described by saying that a uniformly convergent sequence can be integrated term by term.

Proof: We can assume that α is increasing with $\alpha(a) < \alpha(b)$.

To prove (a). we will show that f satisfies Riemann's condition with respect to α on $[a,b]$. (See thm 7.19)

Given $\epsilon > 0$, choose N so that

$$|f(x) - f_N(x)| < \frac{\epsilon}{3[\alpha(b) - \alpha(a)]} \quad \text{for all } x \text{ in } [a,b]$$

Then for every partition P of $[a,b]$ we have.

$|U(P, f - f_N, \alpha)| \le \epsilon/3$ and $|L(P, f - f_N, \alpha)| \le \epsilon/3$

(using the notation of Definition 7.14) for this N, choose P_ϵ so that P finer than P_ϵ implies $U(P, f_N, \alpha) - L(P, f_N, \alpha) < \epsilon/3$

Then for such P we have

$$U(P, f, \alpha) - L(P, f, \alpha) \le U(P, f - f_N, \alpha) - L(P, f - f_N, \alpha) + U(P, f_N, \alpha) - L(P, f_N, \alpha)$$

$$< |U(P, f - f_N, \alpha)| + |L(P, f - f_N, \alpha)| + \epsilon/3 \le \epsilon$$

This proves (a). To prove (b) let $\epsilon > 0$ be given and choose N so that

$$|f_n(t) - f(t)| < \frac{\epsilon}{2[\alpha(b) - \alpha(a)]}$$

for all $n > N$ and every t in $[a, b]$. If $x \in [a, b]$ we have

$$|g_n(x) - g(x)| \le \int_a^x |f_n(t) - f(t)|\, d\alpha(t) \le \frac{\alpha(x) - \alpha(a)}{\alpha(b) - \alpha(a)} \cdot \epsilon/2 < \epsilon$$

This proves that $g_n \to g$ uniformly on $[a, b]$

Theorem: 9.9

Let α be of bounded variation on $[a, b]$ and assume that $\sum f_n(x) = f(x)$ (uniformly on $[a, b]$), where each f_n is a real valued function such that $f_n \in R(\alpha)$ on $[a, b]$, then we have

a) $f \in R(\alpha)$ on $[a, b]$

b) $\int_a^x \sum_{n=1}^{\infty} f_n(t)\, d\alpha(t) = \sum_{n=1}^{\infty} \int_a^x f_n(t)\, d\alpha(t)$ uniformly on $[a, b]$

Proof:

(a) Given: (i) α is of bounded variation on $[a, b]$

(ii) $\sum f_n(x) = f(x)$ (uniformly on $[a, b]$)

(iii) Each f_n is a real valued function such that $f_n \in R(\alpha)$ on $[a, b]$

Consider the partial sum of the series $\sum f_n(x)$

$$s_n(x) = \sum_{k=1}^{n} f_k(x)$$

Since each $f_n \in R(\alpha)$ on $[a, b]$.

$\therefore f_n \in R(\alpha)$ on $[a,b]$

Since $\sum f_n(x) = f(x)$ uniformly on $[a,b]$

$\{s_n(x)\}$ converges uniformly to $f(x)$ on $[a,b]$

$\therefore$ By theorem 9.8

$f \in R(\alpha)$ on $[a,b]$ and

$g_n \to g$ uniformly on $[a,b]$

where $g_n(x) = \int_a^x s_n(t)\, d\alpha(t)$ and

$$g(x) = \int_a^x f(t)\, d\alpha(t) \quad , x \in [a,b]$$

i.e.) $\lim_{n\to\infty} g_n(x) = g(x)$ uniformly on $[a,b]$

i.e.) $\lim_{n\to\infty} \int_a^x s_n(t)\, d\alpha(t) = \int_a^x f(t)\, d\alpha(t)$ uniformly on $[a,b]$

i.e.) $\lim_{n\to\infty} \left\{ \int_a^x \sum_{k=1}^{n} f_k(t)\, d\alpha(t) \right\} = \int_a^x f(t)\, d\alpha(t)$ uniformly on $[a,b]$

i.e.) $\lim_{n\to\infty} \left\{ \sum_{k=1}^{n} \int_a^x f_k(t)\, d\alpha(t) \right\} = \int_a^x f(t)\, d\alpha(t)$ uniformly on $[a,b]$

i.e.) $\sum_{k=1}^{\infty} \int_a^x f_k(t)\, d\alpha(t) = \int_a^x \sum_{n=1}^{\infty} f_n(t)\, d\alpha(t)$ uniformly on $[a,b]$

Note: The conclusion of the above theorem is a uniformly convergent series can be integrated term by term.

9.10 Uniform Convergence and Differentiation:

Theorem 9.13

Assume that each term of $\{f_n\}$ is real-valued function having a finite derivative at each point of an open interval (a,b). assume that for atleast one point x_0 in (a,b) the sequence $\{f_n(x_0)\}$ converges. Assume further that there exists a function g such that $f_n' \to g$ uniformly on (a,b). Then,

(a) There exists a function f such that $f_n \to f$ uniformly on (a,b)

(b) for each x in (a,b) the derivative $f'(x)$ exists and equals $g(x)$

Proof:

given

(i) each term of $\{f_n\}$ is a real valued function having a finite derivative at each point of an open interval (a,b)

(ii) for atleast one point x_0 in (a,b) the sequence

There exists a function g such that $f_n' \to g$ on $[a,b]$

Assume that $c \in (a,b)$ and define a new sequence $\{g_n\}$ as follows

$$g_n(x) = \begin{cases} \dfrac{f_n(x) - f_n(c)}{x-c} & \text{if } x \neq c \\ f_n'(c) & \text{if } x = c \end{cases}$$

Note that the sequence $\{g_n\}$ depends on the choice of c

Since $g_n(c) = f_n'(c)$, and since $\{f_n'\} \to g$ uniformly on (a,b), the sequence $\{g_n(c)\}$ converges — (1)

Now we will prove that $\{g_n\}$ converges uniformly on (a,b)

To prove this we will show that $\{g_n\}$ satisfies cauchy criterion for uniform convergence

To prove this we will show that $\{g_n\}$ satisfies cauchy criterion for uniform convergence

(i) T.P.T for every given $\epsilon > 0$ there exists $N \ni$

$\forall\ n,m > N$ $|g_n(x) - g_m(x)| < \epsilon \quad \forall\ x \in (a,b)$

If $x \neq c$ we have

$$g_n(x) - g_m(x) = \frac{f_n(x) - f_n(c)}{x-c} - \frac{f_m(x) - f_m(c)}{x-c}$$

$$= \frac{[f_n(x) - f_m(x)] - [f_n(c) - f_m(c)]}{x-c}$$

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Let $h(x) = f_n(x) - f_m(x)$

then $g_n(x) - g_m(x) = \dfrac{h(x) - h(c)}{x - c}$ ⟶ (3)

Since $h(x) = f_n(x) - f_m(x)$ and since f_n' exists $\forall$ $x \in (a,b)$ $h'(x)$ exists for each x in (a,b) and

$$h'(x) = f_n'(x) - f_m'(x) \quad \forall\ x \in (a,b) \longrightarrow (4)$$

By applying mean value theorem on the interval with end points c and x we have

$$\frac{h(x) - h(c)}{x - c} = h'(x_1)$$

where x_1 lies between x and c

then (4) $\Rightarrow \dfrac{h(x) - h(c)}{x - c} = f_n'(x_1) - f_m'(x_1)$

(3) $\Rightarrow g_n(x) - g_m(x) = f_n'(x_1) - f_m'(x_1)$ ⟶ (5)

Since $\{f_n'\}$ converges uniformly on (a,b), by Cauchy condition we have,

for any given $\epsilon > 0$ there exists N such that

$$|f_n'(x_1) - f_m'(x_1)| < \epsilon, \quad \forall\ n,m > N.$$

(5) $\Rightarrow |g_n(x) - g_m(x)| = |f_n'(x_1) - f_m'(x_1)| < \epsilon \ \forall\ x \in (a,b)$

$\Rightarrow |g_n(x) - g_m(x)| < \epsilon \quad \forall\ x \in (a,b)$

$\therefore \{g_n\}$ converges uniformly on (a,b)

To prove (a)

we will S.T $\{f_n\}$ converges uniformly on (a,b)

Let us form the particular sequence $\{g_n\}$ corresponding the special point $c = x_0$ for which $\{f_n(x_0)\}$ is assumed to converge

From (1) we have

$$g_n(x) = \frac{f_n(x) - f_n(x_0)}{x - x_0} \quad \forall\ x \in (a,b)$$

$\Rightarrow f_n(x) = f_n(x_0) + (x - x_0)\, g_n(x), \ \forall\ x \in (a,b)$ ⟶ (6)

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$\therefore f_n(x) - f_m(x) = f_n(x_0) + (x-x_0) g_n(x) - f_m(x_0) - (x-x_0) g_m(x)$

$= \{f_n(x_0) - f_m(x_0)\} + (x-x_0)\{g_n(x) - g_m(x)\} \longrightarrow (7)$

Since $\{f_n(x_0)\}$ converges

For any given $\epsilon > 0$ there exists M_1 such that

$\forall\, n, m > M_1$ we have

$$|f_n(x_0) - f_m(x_0)| < \epsilon/2$$

And also since $\{g_n\}$ converges uniformly,

for the same ϵ, we can find M_2 such that $\forall\, n, m > M_2$

$$|g_n(x) - g_m(x)| < \frac{\epsilon}{2|x-x_0|} \longleftrightarrow (8)$$

Let $N = \max(M_1, M_2)$

we have, from (7)

$$|f_n(x) - f_m(x)| = |f_n(x_0) - f_m(x_0) + (x-x_0)(g_n(x) - g_m(x))|$$

$$\leq |f_n(x_0) - f_m(x_0)| + |x-x_0|\,|g_n(x) - g_m(x)|$$

$$< \epsilon/2 + |x-x_0| \frac{\epsilon}{2|x-x_0|}$$

$\Rightarrow |f_n(x) - f_m(x)| < \epsilon/2 + \epsilon/2 = \epsilon$

$|f_n(x) - f_m(x)| < \epsilon \quad \forall\, m, n > N$

which is the cauchy condition for $\{f_n\}$

$\therefore \{f_n\}$ converges uniformly on (a, b)

$\therefore$ There exists a function f on (a,b) such that

$f_n \longrightarrow f$ uniformly on (a, b)

T.P.T for each x in (a,b) the derivative $f'(x)$ exist

and equals $g(x)$

we consider once again the sequence $\{g_n\}$ defined

by (1) for an arbitrary point c in (a,b)

For $x \neq c$, $g_n(x) = \dfrac{f_n(x) - f_n(c)}{x - c}$